POISSON STRUCTURES ON CLOSED MANIFOLDS

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Abstract. We prove an h-principle for poisson structures on closed manifolds.

1. introduction

In this paper we prove an h-principle for poisson structures on closed manifolds. Similar results on open manifolds has been proved by Fernandes and Frejlich in [6]. We state their result below.

Let M^{2n+q} be a C^∞-manifold equipped with a co-dimension-q foliation $\mathcal{F}_0$ and a 2-form ω_0 such that $(\omega_0^n)_{|T\mathcal{F}_0} \neq 0$. Denote by $Fol_q(M)$ the space of co-dimension-q foliations on M identified as a subspace of of $\Gamma(Gr_{2n}(M))$, where $Gr_{2n}(M) \stackrel{pr}{\to} M$ be the grassmann bundle, i.e, $pr^{-1}(x) = Gr_{2n}(T_xM)$ and $\Gamma(Gr_{2n}(M))$ is the space of sections of $Gr_{2n}(M) \stackrel{pr}{\to} M$ with compact open topology. Define

$$\Delta_q(M) \subset Fol_q(M) \times \Omega^2(M)$$

$$\Delta_q(M) := \{(\mathcal{F}, \omega) : \omega^n_{|T\mathcal{F}}\} \neq 0$$

Obviously $(\mathcal{F}_0, \omega_0) \in \Delta_q(M)$. Inthis setting Fernandes and Frejlich has proved the following

Theorem 1.1. *([6]) Let M^{2n+q} be an open manifold with $(\mathcal{F}_0, \omega_0) \in \Delta_q(M)$ be given. Then there exists a homotopy $(\mathcal{F}_t, \omega_t) \in \Delta_q(M)$ such that ω_1 is $d_{\mathcal{F}_1}$-closed (actually exact).*

In the language of poisson geometry the above result 1.1 takes the following form. Let $\pi \in \Gamma(\wedge^2 TM)$ be a bi-vectorfield on M, define $\#\pi : T^*M \to TM$ as $\#\pi(\eta) = \pi(\eta, -)$. If $Im(\#\pi)$ is a regular distribution then π is called a regular bi-vectorfield.

Theorem 1.2. *Let M^{2n+q} be an open manifold with a regular bi-vectorfield π_0 on it such that $Im(\#\pi)$ is an integrable distribution then π_0 can be homotoped through such bi-vectorfields to a poisson bi-vectorfield π_1.*

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In 1.1 above $d_{\mathcal{F}}$ is the tangential exterior derivative, i.e, for $\eta \in \Gamma(\wedge^k T^*\mathcal{F})$, $d_{\mathcal{F}}\eta$ is defined by the following formula

$$d_{\mathcal{F}}\eta(X_0, X_1, ..., X_k) = \Sigma_i(-1)^i X_i(\eta(X_0, .., \hat{X}_i, .., X_k))$$

$$+\Sigma_{i<j}(-1)^{i+j}\eta([X_i, X_j], X_0, .., \hat{X}_i, .., \hat{X}_j, .., X_k)$$

where $X_i \in \Gamma(T\mathcal{F})$. So if we extend a $\mathcal{F}$-leafwise closed k-form η, i.e, $d_{\mathcal{F}}\eta = 0$, to a form η' by the requirement that $ker(\eta') = \nu\mathcal{F}$, where $\nu\mathcal{F}$ is the normal bundle to $\mathcal{F}$, then $d\eta' = 0$.

In order to fix the foliation in 1.1 one needs to impose an openness condition on the foliation, we refer the readers to [1] for precise definition of this openness condition. Under this hypothesis Bertelson proved the following

Theorem 1.3. *([1]) If $(M, \mathcal{F})$ be an open foliated manifold with $\mathcal{F}$ satisfies some openness condition and let ω_0 be a $\mathcal{F}$-leaf wise 2-form then ω_0 con be homotoped through $\mathcal{F}$-leaf wise 2-forms to a $\mathcal{F}$-leaf wise symplectic form.*

She also constructed counter examples in [2] that without this openness condition the above theorem fails. A contact analogue of Bertelson's result on any manifold (open or closed) has recently been proved in [3] by Borman, Eliashberg and Murphy. We have used this theorem in our argument. So let us state the theorem.

Theorem 1.4. *([3]) Let M^{2n+q+1} be any manifold equipped with a co-dimension-q foliation $\mathcal{F}$ on it and let $(\alpha_0, \beta_0) \in \Gamma(T^*\mathcal{F} \oplus \wedge^2 T^*\mathcal{F})$ be given such that $\alpha_0 \wedge \beta_0^n$ is nowhere vanishing, then there exists a homotopy $(\alpha_t, \beta_t) \in \Gamma(T^*\mathcal{F} \oplus \wedge^2 T^*\mathcal{F})$ such that $\alpha_t \wedge \beta_t^n$ nowhere vanishing and $\beta_1 = d_{\mathcal{F}}\alpha_1$.*

Now we state the main theorem of this paper.

Theorem 1.5. *Let M^{2n+q} be a closed manifold with $q = 2$ and $(\mathcal{F}_0, \omega_0) \in \Delta_q(M)$ be given. Then there exists a homotopy $\mathcal{F}_t$ of singular foliations on M with singular locus Σ_t and a homotopy of two forms ω_t such that the restriction of (ω_t) to $T\mathcal{F}_t$ is non-degenerate and ω_1 is closed.*

In terms of poisson geometry 1.5 states

Theorem 1.6. *Let M^{2n+q} be a closed manifold with $q = 2$ and π_0 be a regular bi-vectorfield of rank $2n$ on it such $Im(\#\pi_0)$ is integrable distribution. Then there exists a homotopy of*

bi-vectorfields $\pi_t,\ t \in I$ (not regular) such that $Im(\#\pi_t)$ integrable and π_1 is a poisson bi-vectorfield.

We organize the paper as follows. In section-2 we shall explain the preliminaries of the theory of h-principle and of wrinkle maps which are needed in the proof of 1.5 which we present in section-3.

2. Preliminaries

We begin with the theory of h-principle. Let $X \to M$ be any fiber bundle and let $X^{(r)}$ be the space of r-jets of jerms of sections of $X \to M$ and $j^r f : M \to X^{(r)}$ be the r-jet extension map of the section $f : M \to X$. A section $F : M \to X^{(r)}$ is called holonomic if there exists a section $f : M \to X$ such that $F = j^r f$. In the following we use the notation $Op(A)$ to denote a small open neighborhood of $A \subset M$ which is unspecified.

Theorem 2.1. *([4]) Let $A \subset M$ be a polyhedron of positive co-dimension and $F_z : Op(A) \to X^{(r)}$ be a family of sections parametrized by a cube $I^m,\ m = 0, 1, 2, ...$ such that F_z is holonomic for $z \in Op(\partial I^m)$. Then for given small $\varepsilon, \delta > 0$ there exists a family of δ-small (in the C^0-sense) diffeotopies $h_z^\tau : M \to M,\ \tau \in [0,1],\ z \in I^m$ and a family of holonomic sections $\tilde{F}_z : Op(h_z^1(A)) \to X^{(r)},\ z \in I^m$ such that*

1. *$h_z^\tau = id_M$ and $\tilde{F}_z = F_z$ for all $z \in Op(\partial I^m)$*

2. *$dist(\tilde{F}_z(x), (F_z)_{|Oph_z^1(A)}(x)) < \varepsilon$ for all $x \in Op(h_z^1(A))$*

Remark 2.2. *Relative version of 2.1 is also true. More precisely let the sections F_z be already holonomic on $Op(B)$ for a sub-polyhedron B of A, then the diffeotopies h_z^τ can be made to be fixed on $Op(B)$ and $\tilde{F}_z = F_z$ on $Op(B)$.*

Now we briefly recall preliminaries of wrinkled maps following [5]. Consider the following map

$$w : \mathbb{R}^{q-1} \times \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{q-1} \times \mathbb{R}$$

$$w_s(y, x, z) = (y, z^3 + 3(|y|^2 - 1)z - \Sigma_1^s x_i^2 + \Sigma_{s+1}^{2n} x_i^2)$$

where $y \in \mathbb{R}^{q-1}$, $z \in \mathbb{R}$ and $x \in \mathbb{R}^{2n}$. Observe that the singular locus of w_s is

$$\Sigma(w_s) = \{x = 0,\ z^2 + |y|^2 = 1\}$$

Let D be the disc enclosed by $\Sigma(w_s)$, i.e,

$$D = \{x = 0,\ z^2 + |y|^2 \leq 1\}$$

Definition 2.3. *([5]) A map $f : M \to Q$ is called a wrinkled map if there exists a disjoint union of open subsets $U_1, ..., U_l \subset M$ such that $f_{|M-U}$ is a submersion, where $U = \cup_1^l U_i$ and $f_{|U_i}$ is equivalent to w_s, for some s.*

A fibered map over B is given by a map $f : U \to V$, where $U \subset M$ and $V \subset Q$ with submersions $a : U \to B$ and $b : V \to B$ such that $b \circ f = a$.
Denote by $T_B M$ and $T_B Q$, $ker(a) \subset TM$ and $ker(b) \subset TQ$ respectively. The fibered differential $df_{|T_B M}$ is denoted by $d_B f$.
If we consider the projection on first k-factors, where $k < q - 1$, then w_s is a fibered map. So we can define fibered version of a wrinkled map. We refer the reader [5] for more details.By combining Lemma-2.1B and Lemma-2.2B of [5] we get the following

Theorem 2.4. *([5]) Let $g : I^n \to I^q$ be a fibered submersion over I^k and $\theta : I^n \to I^n$ be a fibered wrinkled map over I^k with one wrinkle. Then there exists a fibered wrinkled map ψ with very small wrinkles and which agrees with θ near ∂I^n such that $g \circ \psi$ is a fibered wrinkled map.*

3. Main Theorem

In this section we prove 1.5.

Consider $\tilde{M} = M \times \mathbb{R}$ and let us denote the co-dimension-q foliation $\mathcal{F}_0 \times \mathbb{R}$ on $\tilde{M}$ by $\tilde{\mathcal{F}}$ with a $\tilde{\mathcal{F}}$-leaf wise one form α_0 such that $\alpha_0(\partial_s) = 1$ and $ker(\alpha_0)_{|(x,s)} = T_x\mathcal{F}_0$. Observe that if we extend ω_0 to $\tilde{M}$ by the requirement that $\omega_0(\partial_s, -) = 0$, then $(\alpha_0 \wedge \omega_0^n)_{|T\tilde{\mathcal{F}}} \neq 0$. Let $(\omega_0)_{|T\tilde{\mathcal{F}}} = \beta_0$. Then (α_0, β_0) is a $\tilde{\mathcal{F}}$-leaf wise almost contact structure. Then according 1.4 there exists a homotopy of pairs (α_t, β_t) defining a homotopy of $\tilde{\mathcal{F}}$-leaf-wise almost contact structures consisting of a $\tilde{\mathcal{F}}$-leaf-wise one form α_t and a $\tilde{\mathcal{F}}$-leaf-wise two form β_t such that $\beta_1 = d_{\tilde{\mathcal{F}}}\alpha_1$, i.e, (α_1, β_1) is a $\tilde{\mathcal{F}}$-leaf-wise contact structure. Now let $L_t = ker(\alpha_t) \subset T\tilde{\mathcal{F}}$ and $G_t^1 = L_t \oplus \nu\tilde{\mathcal{F}} \oplus \mathbb{R} \subset \tilde{M} \times \mathbb{R}$, where $\nu\tilde{\mathcal{F}}$ is the normal bundle.

Now observe that the embedding $f_0 : M \to M \times \{0\} \times \{1\} \hookrightarrow \tilde{M} \times \mathbb{R}$ is $\pitchfork$ to $\tilde{\mathcal{F}} \times \mathbb{R}$ and $Im(df_0) \cap (T\tilde{\mathcal{F}} \times \mathbb{R}) = L_0$. First extend β_t to $\tilde{M}$ and call it $\tilde{\beta}_t$ in such a way that $ker(\tilde{\beta}_t) = \nu\tilde{\mathcal{F}}$. Let $X_t = ker(\beta_t)$ be the vector field on $\tilde{M}$ and consider the family of 2-dimensional foliation $\mathcal{G}_t$ generated by X_t and ∂_w, where w is the $\mathbb{R}$-variable in $\tilde{M} \times \mathbb{R}$. Observe that $\alpha_t \wedge dw$ is a

$\mathcal{G}_t$-leaf-wise symplectic form.

Now we shall perturb f_0 by a homotopy of immersions f_t such that f_t will be tangent to $\tilde{\mathcal{F}} \times \mathbb{R}$ only on Σ_t and on $M - \Sigma_t$, $f_t \pitchfork \tilde{\mathcal{F}} \times \mathbb{R}$, i.e, $Im(df_t) \cap (T\tilde{\mathcal{F}} \times \mathbb{R})$ is of dimension $2n$ and $Im(df_t) \cap (T\tilde{\mathcal{F}} \times \mathbb{R})$ is close to L_t. As $\tilde{\beta}^n_{t|L_t} \neq 0$, we conclude that the restriction of $w\tilde{\beta}_t + \alpha_t \wedge dw$ is non-degenerate on $Im(df_t) \cap T\tilde{\mathcal{F}} \times (0, \infty)$. Hence we only need to set $\mathcal{F}_t = f_t^{-1}(\tilde{\mathcal{F}} \times (0, \infty))$ and $\omega_t = f_t^*(w\tilde{\beta}_t + \alpha_t \wedge dw)$.

First divide the interval I as

$$I = \cup_1^N [(i-1)/N, i/N]$$

and assume that f_t is defined on $[0, (i-1)/N]$. Observe that the limit

$$lim_{x \to \Sigma_{(i-1)/N}} Im(df_{(i-1)/N}) \cap (T\tilde{\mathcal{F}} \times \mathbb{R})$$

exists and is of dimension $2n$ and is close to $L_{(i-1)/N}$. Let $\bar{L}_{(i-1)/N} \subset T\tilde{\mathcal{F}} \times \mathbb{R}$ be the $2n$-dimensional distribution which equals $Im(df_{(i-1)/N}) \cap T\tilde{\mathcal{F}} \times \mathbb{R}$ on $M - \Sigma_{(i-1)/N}$ and on $\Sigma_{(i-1)/N}$ it is the limit. Set $\nu_{(i-1)/N} = Im(df_{(i-1)/N})/\bar{L}_{(i-1)/N}$ and $G^i_t,\ t \in [(i-1)/N, i/N]$ as

$$G^i_t = L_t \oplus \nu_{(i-1)/N}$$

Observe that $Im(df_{(i-1)/N})$ approximates $G^i_{(i-1)/N}$. So if N is large then there exists a family of monomorphisms $F_t,\ t \in [(i-1)/N, i/N]$ such that $F_{(i-1)/N} = df_{(i-1)/N}$ and $Im(F_t)$ approximates G^i_t and hence F_t tangent to $T\tilde{F} \times \mathbb{R}$ only on a slightly perturbed $\Sigma_{(i-1)/N}$.

Choose a triangulation of M which is fine and $\Sigma_{(i-1)/N} \subset A$, where A is the $(2n+q-1)$-skeleton of the triangulation. As the triangulation is fine all $(2n+q)$-simplices under the image of $f_{(i-1)/N}$ is contained in a neighborhood diffeomorphic to I^{2n+q+2} and on it $\tilde{\mathcal{F}} \times \mathbb{R}$ is given by the projection $\pi : I^{2n+q+2} \to I^q$ (projection on the first q-factors).

Without loss of generalization let us assume F_t is defined for $t \in I$ instead of $t \in [(i-1)/N, i/N]$. Let

$$\bar{F}_t = F_{\sigma(t)}$$

where $\sigma : I \to I$ is a smooth map such that $\sigma = 0$ on $[0, \varepsilon] \cup [1-\varepsilon, 1]$ and $\sigma = 1$ on a neighborhood of $1/2$.

Use 2.1 for $\bar{F}_t$ to get a family of immersions $\bar{f}_t$ defined on $Op(h_t^1(A))$ and approximating $\bar{F}_t$ on $Op(h_t^1(A))$, where h_t^τ is δ small with $h_t^1 = id$ for $t \in [0, \varepsilon] \cup [1-\varepsilon, 1]$. The δ above will be used later so the reader needs to keep note of this fact. We approximate $\bar{F}_t$ by F'_t such that $F'_t = d\bar{f}_t$ on $Op(h_t^1(A))$.

It is enough to consider one simplex Δ. Let $\Delta' \subset \Delta$ be a 2δ-smaller simplex so that $h_t^1(\Delta)$ does not intersect Δ', δ is produced by applying 2.1 to $\bar{F}_t$ above.

Define monomorphisms $\tilde{F}_t^\delta$ depending on δ as follows. On $Op(\partial\Delta)$, set $\tilde{F}_t^\delta = d(\bar{f}_t \circ h_t^1)$. Now observe there exists an isotopy of embeddings

$$\tilde{g}_\tau : \Delta - Op(\partial\Delta) \to \Delta - Op(\partial\Delta)$$

such that $\tilde{g}_0 = id$ and $\tilde{g}_1(\Delta - Op(\partial\Delta)) = \Delta'$. Any element of $(\Delta - Op(\partial\Delta)) - \Delta'$ is of the form $\tilde{g}_\tau(x),\ x \in \partial(\Delta - Op(\partial\Delta))$.

Let $\gamma_t^x : I \to M$ be the path

$$\begin{aligned} \gamma_t^x(\tau) &= h_t^{1-2\tau}(x),\ \tau \in [0, 1/2] \\ &= \tilde{g}_{2\tau-1}(x),\ \tau \in [1/2, 1] \end{aligned}$$

Set $(\tilde{F}_t^\delta)_{\tilde{g}_\tau(x)} = (F'_t)_{\gamma_t^x(\tau)}$. Observe that $\gamma_t^x(1) = \tilde{g}_1(x) \in \partial\Delta'$. As $\tilde{F}_t^\delta$-agrees with F'_t along $\partial\Delta'$, we can extend $\tilde{F}_t^\delta$ on δ by defining it to be F'_t on Δ'. Observe that

$$\Sigma_t^\delta = \{\tilde{\mathcal{F}}_t^\delta \textit{ tangent to } T\tilde{\mathcal{F}} \times \mathbb{R}\} \subset \Delta - \Delta'$$

The next theorem 3.1 extends f_t from $t \in [0, (i-1)/N]$ to $t \in [0, i/N]$. To start the process i.e, to extend f_0 to $f_t,\ t \in [0, 1/N]$ we take a fine triangulation of M so that image under f_0 of all top dimensional simplices lies in a neighborhood diffeomorphic to I^{2n+q+2} and on it $\tilde{\mathcal{F}} \times \mathbb{R}$ is given by the projection on the first q factors $\pi : I^{2n+q+2} \to I^q$.

Theorem 3.1. *Let $I_\delta = [\delta, 1-\delta]$, $I_\varepsilon = [\varepsilon, 1-\varepsilon]$ with $\varepsilon = \varepsilon(\delta) < \delta$ and $(F_t^\delta, b_t^\delta) : TI^{2n+q} \to TI^{2n+q+2}$ be a family of monomorphisms such that*

(1) $F_t^\delta = db_t^\delta$ *on* $I^{2n+q} - I^{2n+q}_{\varepsilon(\delta)}$

(2) *$F^\delta_t \pitchfork \mathcal{L}$ on I^{2n+q}_δ for all t and $Im(F^\delta_t) \cap T\mathcal{L}$ is of dimension $2n$ and is close to L_t for all t on I^{2n+q}_δ*

(3) *$\Sigma^\delta_t = \{F^\delta_t$ tangent to $T\mathcal{L}\} \subset (I^{2n+q} - I^{2n+q}_\delta)$*

where $\mathcal{L}$ is the foliation on I^{2n+q+2} induced by the projection $\pi : I^{2n+q+2} \to I^q$ (projection on the first q-factors), $\tilde{\mathcal{L}}$ is such that $\mathcal{L} = \tilde{\mathcal{L}} \times I$ and $L_t \subset T\tilde{\mathcal{L}}$ is a family of $2n$-dimensional distribution. Then there is a δ'' and a family of immersions $f_t : I^{2n+q} \to I^{2n+q+2}$ such that

(1) *$f_t = b^{\delta''}_t$ on $I^{2n+q} - I^{2n+q}_{\varepsilon(\delta'')/2}$*

(2) *$(\pi \circ f_t)_{|I^{2n+q}_{\delta''}}$ is a wrinkle map*

(3) *If $\Sigma_t(I^{2n+q} - I^{2n+q}_{\delta''}) = \{x \in I^{2n+q} - I^{2n+q}_{\delta''} : f_t(x)$ tangent to $\mathcal{L}\}$, then on $(I^{2n+q} - I^{2n+q}_{\delta''}) - \Sigma_t(I^{2n+q} - I^{2n+q}_{\delta''})$, $Im(df_t) \cap T\mathcal{L}$ is of dimension $2n$ and is close to L_t.*

Proof. Let $\sigma : I \to I$ be a smooth map such that $\sigma = 0$ on $I - I_{\varepsilon(\delta)}$ and $\sigma = 1$ on a neighborhood of $1/2$. Let

$$F^\delta : T(I \times I^{2n+q}) \to T(I \times I^{2n+q+2})$$

be monomorphisms given by the matrix

$$F^\delta_{(t,x)} = \begin{pmatrix} 1 & 0 \\ \partial_t b^\delta_{\sigma(t)}(x) & F^\delta_{\sigma(t)}(x) \end{pmatrix}$$

Which covers $b^\delta(t,x) = (t, b^\delta_{\sigma(t)}(x))$. So $F^\delta = db^\delta$ on $I \times (I^{2n+q} - I^{2n+q}_{\varepsilon(\delta)})$. Let $\chi^\delta : I^{2n+q+1} \to I$ be a smooth map such that $\chi^\delta = 0$ on $I^{2n+q+1} - I^{2n+q+1}_{\varepsilon(\delta)}$ and $\chi^\delta = 1$ on $I^{2n+q+1}_{\delta'}$, $\delta' < \delta$. Set $\Xi_\tau : I^{2n+q} \to I^{2n+q}$, $\tau \in I$ as

$$\Xi_\tau(x_1, ..., x_{2n+q}) = (x_1, ..., x_{q-1}, (1-\chi^\delta)x_q + \chi^\delta(\tau - \gamma'(\tau).x_q), x_{q+1}, ..., x_{2n+q})$$

where $\gamma' : I \to [-\varepsilon(\delta)/2, \varepsilon(\delta)/2]$ be linear homeomorphism such that $\gamma'(0) = -\varepsilon(\delta)/2$ and $\gamma'(1) = \varepsilon(\delta)/2$. Now set $(F^\delta_\tau)_{(t,x)} = F^\delta_{(t,\Xi_\tau(x))}$ which covers $b^\delta_\tau(t,x) = b^\delta(t, \Xi_\tau(x))$. Observe that

(1) $F^\delta_\tau = db^\delta = db^\delta_\tau$ on $I_{\varepsilon(\delta)} \times I^{q-1}_{\varepsilon(\delta)} \times I \times I^{2n}_{\varepsilon(\delta)}$

(2) $F_0^\delta = db^\delta = db_\tau^\delta$ on $I \times I^{q-1} \times [0, \varepsilon(\delta)] \times I^{2n}$

(3) $F_1^\delta = db^\delta = db_\tau^\delta$ on $I \times I^{q-1} \times [1-\varepsilon(\delta), 1] \times I^{2n}$

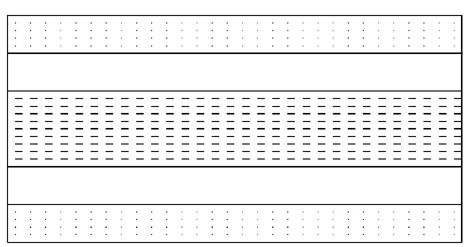
Moreover observe that F_0^δ and F_1^δ are holonomic and for $\tau \in I_\delta$

$$\Sigma_\tau^\delta = \{F_\tau^\delta \ \textit{not} \ \pitchfork \ \textit{to} \ T\mathcal{L} \times \mathbb{R}^2\} \subset I^{2n+q+1} - I_\delta^{2n+q+1}$$

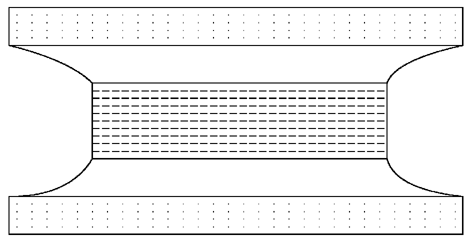
Using the 2.1 we can approximate F_τ^δ on $h_\tau^1(I \times I^{q-1} \times \{1/2\} \times I^{2n})$ by df_τ^δ, where f_τ^δ is a family of immersions defined on $h_\tau^1(I \times I^{q-1} \times \{1/2\} \times I^{2n})$. Now consider three smooth functions $\chi^i, \ i = 1, 2, 3$ defined as follows
$\chi^1 : [0, \delta'] \to [0, 1]$, $\chi^1(0) = 0$ and $\chi^1(\delta') = 1$. $\chi^2 : [1-\delta', 1] \to [0, 1]$, $\chi^2(1-\delta') = 1$ and $\chi^2(1) = 0$. $\chi^3 : [\delta', 1-\delta'] \to [0, 1]$, $\chi^3(\delta') = 0$ and $\chi^3(1-\delta') = 1$ also $\chi^3(\delta) = \delta$ and $\chi^3(1-\delta) = 1-\delta$. Now define g_τ^δ as follows

$$\begin{aligned} g_\tau^\delta &= b_0^\delta \circ g_{\chi^1(\tau)}, \ \tau \in [0, \delta'] \\ &= f_{\chi^3(\tau)}^\delta \circ h_{\chi^3(\tau)}^1 \circ g_1, \ \tau \in [\delta', 1-\delta'] \\ &= b_1^\delta \circ g_{\chi^2(\tau)}, \ \tau \in [1-\delta', 1] \end{aligned}$$

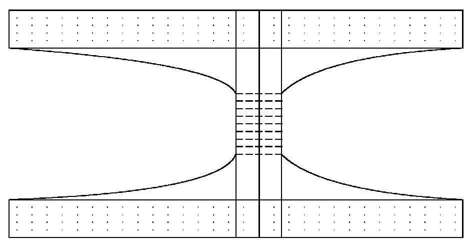
Where $g_s : I \times I^{2n+q} \to I \times I^{2n+q+2}, \ s \in I$ is an isotopy of embeddings defined as follows



$I \times I^{2n+q} = g_0(I \times I^{2n+q})$



$g_s(I \times I^{2n+q})$



$g_1(I \times I^{2n+q})$

$g_s = id,\ s \in I$ on $(I - I_{\varepsilon(\delta)/2}) \times (I - I_{\varepsilon(\delta)/2})^{q-1} \times I \times (I - I_{\varepsilon(\delta)/2})^{2n}$. This is shown as shaded region at the top and bottom in the pictures above.

Let $\bar{g}_s : I \to I$ be such that $\bar{g}_0 = id$ and $\bar{g}_1(I) \subset Op(1/2)$. Then we set $g_s = id_{I_{\varepsilon(\delta)}} \times id_{I^{q-1}_{\varepsilon(\delta)}} \times \bar{g}_s \times id_{I^{2n}_{\varepsilon(\delta)}}$ on $I_{\varepsilon(\delta)} \times I^{q-1}_{\varepsilon(\delta)} \times I \times I^{2n}_{\varepsilon(\delta)}$. This is shown in the central shaded region in the above pictures.

In the non-shaded region in the third picture i.e, in the picture of $g_1(I^{2n+q+1})$,

$$f_0^\delta = b_0^\delta = b_\tau^\delta = b_1^\delta = f_1^\delta$$

and hence g_τ^δ is well defined.

Now observe that for $\tau \in I_\delta$, $\{g_\tau^\delta \ \textit{not} \ \pitchfork \ \textit{to} \ \mathcal{L} \times \mathbb{R}\} \subset (I^{2n+q+1} - I_\delta^{2n+q+1})$.

For an integer $l > 0$ take a function $\phi_l : I \to I$ such that

$$\begin{aligned} \phi_l &= 1, \ on \ I_{1/(8l)} \\ &= 0, \ outside \ I_{1/(16l)} \end{aligned}$$

which is increasing on $[1/(16l), 1/(8l)]$ and decreasing on $[1-1/(8l), 1-1/(16l)]$. Set

$$\gamma_l(t) = t + \phi_l(t) Sin(2\pi l t), \ t \in I$$

Let J_i be the interval of length $9/(16l)$ centered at $(2i-1)/2l$. Observe that γ_l is non-singular outside $\cup J_i$ and $(\gamma_l)_{J_i}$ is a wrinkle. Also

$$\partial_t \gamma_l(t) \geq l, \ t \in I - \cup J_i$$

Let $\bar{\chi}^\delta : I^{2n+q+1} \to I$ be such that

$$\begin{aligned} \bar{\chi}^\delta &= 0, \ near \ \partial(I^{2n+q+1}) \\ &= 1, \ on \ I^{2n+q+1}_{\varepsilon(\delta)} \end{aligned}$$

Now we take $\delta = \delta(l) << 1/(16l)$. Set $\tilde{\gamma}_l(x) = (1-\bar{\chi}^\delta(x))x_q + \bar{\chi}^\delta(x)\gamma_l(x_q)$. Let $\lambda : I \to I$, be such that $\lambda(0) = 0$, $\lambda(1) = 1$

(1) $\lambda = (2i-1)/2l, \ on \ J_i$

(2) $0 < \partial_t \lambda < 3, \ on \ I - \cup J_i$

Set $\bar{g}^\delta_\tau = g^\delta_{\lambda(\tau)}, \ \tau \in I$. Now consider

$$(t, x_1, ..., x_{2n+q}) \xrightarrow{\rho_l} \bar{g}^\delta_{x_q}(t, x_1, ..., x_{q-1}, \tilde{\gamma}_l(x), x_{q+1}, ..., x_{2n+q})$$

Let θ be the function $\theta(t,x) = (t, x_1, ..., x_{q-1}, \tilde{\gamma}_l(x), x_{q+1}, ..., x_{2n+q})$. Then θ is a wrinkle map and as $\delta = \delta(l) << 1/(16l)$, the wrinkles of θ do not intersect $\{g^\delta_\tau \ not \ \pitchfork \ to \ \mathcal{L} \times \mathbb{R}\}$, for $\tau \in I_\delta$. On $I \times I^{q-1} \times J_i \times I^{2n}$, ρ_l is of the form $\bar{g}_i^{\delta(l)} \circ \theta_i$, where $\theta_i = \theta_{|I \times I^{q-1} \times J_i \times I^{2n}}$. So using 3.1 we can replace θ_i by another wrinkle map ψ_i such that $\pi \circ \bar{g}_i^{\delta(l)} \circ \psi_i$ turns out to be a fibered wrinkle map, fibered over the first factor I. But observe that $\bar{g}_i^{\delta(l)} \circ \psi_i$ is not an immersion. So we need to regularize it.

For all i, $\pi \circ \bar{g}_i^{\delta(l)} \circ \psi_i$ has many wrinkles and near each wrinkle it is of the form

$$w_s(t, y, z, x) = (t, y, z^3 + 3(|(t,y)|^2 - 1)z - \Sigma_1^s x_i^2 + \Sigma_{s+1}^{2n} x_i^2)$$

and hence $\bar{g}_i^{\delta(l)} \circ \psi_i$ is of the form

$$(t,y,z,x) \mapsto (t,y,z^3+3(|(t,y)|^2-1)z-\Sigma_1^s x_i^2+\Sigma_{s+1}^{2n}x_i^2, a_1(t,y,z,x),...,a_{2n+2}(t,y,z,x))$$

Its derivative is given by the matrix

$$\begin{pmatrix} I_q & 0 & 0 \\ * & 3(z^2+|(t,y)|^2-1) & (\pm 2x_i)_1^{2n} \\ * & (\partial_z a_j)_1^{2n+2} & (\partial_{x_i} a_j)_{i=1,j=1}^{i=2n,j=2n+2} \end{pmatrix}$$

and from the proof of 3.1 in [5] it follows that $\partial_z a_j = 0$ for all j along $\{z^2+|(t,y)|^2-1=0\}$. So in order to regularize it one needs to C^1-approximate a_j's by a'_j's so that not all of $\partial_z a'_j$ vanish simultaneously along $\{z^2+|(t,y)|^2=1\}$. But we shall moreover want the $\partial_z a'_{2n+1} \neq 0$ along $\{z^2+|(t,y)|^2-1=0\}$, where a'_{2n+1} corresponds to the $\mathbb{R}$-factor of $\tilde{M}=M\times\mathbb{R}$.

Now let us set $\varphi : I^{2n+q+2} \to [0,1]$ be a smooth function such that $\varphi=1$ outside a neighborhood of D, where D is the disc which encloses $\{z^2+|(t,y)|^2-1=0\}$ and on $\{z^2+|(t,y)|^2-1=0\}$, $\varphi=0$ and $\partial_{y_1}\varphi=0$, moreover $\phi+y_1\partial_{y_1}\varphi$ is non-vanishing outside $\{z^2+|(t,y)|^2-1=0\}$. Now let $y=(y_1,...,y_q)$ in the above. Now replace the resulting map by

$$(t,y,z,x) \mapsto (t,\varphi(t,y,z,x)y_1,y_2,...,y_q,z^3+3(|(t,y)|^2-1)z-\Sigma_1^s x_i^2$$
$$+\Sigma_{s+1}^{2n}x_i^2, a'_1(t,y,z,x),...,a'_{2n+2}(t,y,z,x)+y_1-y_1\varphi(t,y,x,z))$$

Where in the above the last component corresponds to the $\mathbb{R}$-component of $\tilde{M}\times\mathbb{R}$, i.e, the w-variable. Its derivative is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \partial_t(y_1\varphi) & \varphi+y_1\partial_{y_1}\varphi & * & * & * \\ 0 & 0 & I_{q-2} & 0 & 0 \\ * & * & * & 3(z^2+|(t,y)|^2-1) & (\pm 2x_i)_1^{2n} \\ * & * & * & (\partial_z a'_j)_1^{2n+1} & (\partial_{x_i} a'_j)_{i=1,j=1}^{i=2n,j=2n+1} \\ *-\partial_t(y_1\varphi) & *+1-\partial_{y_1}(y_1\varphi) & *-\partial_{y_k}(y_1\varphi) & (\partial_z a'_{2n+2}-\partial_z(y_1\varphi)) & (\partial_{x_i}a'_{2n+2}-\partial_{x_i}(y_1\varphi))_{i=1}^{i=2n} \end{pmatrix}$$

Now observe that the projections of the column vectors

$$(0,*,0,3(z^2+|(t,y)|^2-1),(\partial_z a'_j)_1^{2n+1},(\partial_z a'_{2n+2}-\partial_z(y_1\varphi)))^T$$

and

$$(0,*,0,(\pm 2x_i)_1^{2n},(\partial_{x_i}a'_j)_{i=1,j=1}^{i=2n,j=2n+1},(\partial_{x_i}a'_{2n+2}-\partial_{x_i}(y_1\varphi))_{i=1}^{i=2n})^T$$

onto $T\tilde{\mathcal{F}} \times \mathbb{R}$ are

$$((\partial_z a'_j)_1^{2n+1}, (\partial_z a'_{2n+2} - \partial_z(y_1\varphi)))^T$$

and

$$((\partial_{x_i} a'_j)_{i=1,j=1}^{i=2n,j=2n+1}, (\partial_{x_i} a'_{2n+2} - \partial_{x_i}(y_1\varphi))_{i=1}^{i=2n})^T$$

and their projection on $T\tilde{\mathcal{F}}$ are

$$((\partial_z a'_j)_1^{2n+1})^T$$

and

$$((\partial_{x_i} a'_j)_{i=1,j=1}^{i=2n,j=2n+1})^T$$

Whose span was already close to $\mathbb{R} \times L_t$.

Along $\{g_\tau^\delta \textit{ not } \pitchfork \textit{ to } \mathcal{L} \times \mathbb{R}\} \subset I^{2n+q+1} - I_\delta^{2n+q+1}$, $\tau \in I_\delta$, we can apply the same technique as above. For this we decompose $\{g_\tau^\delta \textit{ not } \pitchfork \textit{ to } \mathcal{L} \times \mathbb{R}\} \subset I^{2n+q+1} - I_\delta^{2n+q+1}$, $\tau \in I_\delta$ as

$$\{g_\tau^\delta \textit{ not } \pitchfork \textit{ to } \mathcal{L} \times \mathbb{R}\} = \{\partial_{y_1} \textit{ tangent to } \mathcal{L} \times \mathbb{R}\} \cup \{\partial_{y_2} \textit{ tangent to } \mathcal{L} \times \mathbb{R}\}$$

Now we use the same technique as above along $\{\partial_{y_1} \textit{ tangent to } \mathcal{L} \times \mathbb{R}\}$ and $\{\partial_{y_2} \textit{ tangent to } \mathcal{L} \times \mathbb{R}\}$, i.e, rotating y_1-component to be tangent to $\mathcal{L} \times \mathbb{R}$ along $\{\partial_{y_2} \textit{ tangent to } \mathcal{L} \times \mathbb{R}\}$ and same for y_2 along $\{\partial_{y_1} \textit{ tangent to } \mathcal{L} \times \mathbb{R}\}$. This way we make $\{g_\tau^\delta \textit{ not } \pitchfork \textit{ to } \mathcal{L} \times \mathbb{R}\}$ to $\{g_\tau^\delta \textit{ tangent to } \mathcal{L} \times \mathbb{R}\}$.

Note that if $q > 2$, then along intersection of three sets $\cap_{i=1}^3 \{\partial_{y_i} \textit{ tangent to } \mathcal{L} \times \mathbb{R}\}$, we can not make $\{g_\tau^\delta \textit{ not } \pitchfork \textit{ to } \mathcal{L} \times \mathbb{R}\}$ to $\{g_\tau^\delta \textit{ tangent to } \mathcal{L} \times \mathbb{R}\}$, otherwise the rank will drop and it will no longer be regular.

Let $\bar{\rho}_l$ be the regularized map, then $\bar{\rho}_l$ is of the form $\bar{\rho}_l(t, x) = (t, x(t))$, where $x(t)$ are functions of t. So the required family of immersions is given by

$$f_t(x) = x(\sigma^{-1}(t)),\ t \in [0, 1/2]$$

with reparametrization. Clearly f_t has the property (1) *and* (2). Condition (3) follows from the fact that for large l, $d_I\rho_l$ approximates $d_I\bar{g}_\tau^\delta$ on $I \times I^{q-1} \times (I - \cup_i J_i) \times I^{2n}$ and on $I^{2n+q+1} - I_{\delta(l)}^{2n+q+1}$ whose proof is same as in 2.3A of [5] and we refer the readers to [5]. As $\delta(l)$ depends on l and $\varepsilon(\delta)$ depends on δ, we are done. $\square$

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